SEPARABLE BANACH SPACES WHICH ADMIT l_n^∞ APPROXIMATIONS

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ABSTRACT

In this paper we study a class of separable Banach spaces which can be approximated by certain special finite-dimensional subspaces. This class is characterized in Theorem 1.1, from which it follows that the space of continuous scalar-valued functions on a compact metric space always belongs to this class, and that every member of this class has a monotone basis.

1. **Introduction.** In several recent papers [7, p. 25], [2], [5], [8], the concept of a π_1-space has been found useful: A Banach space B is a *π_1-space* if it contains a directed (by inclusion) family $(E_\alpha)_{\alpha \in A}$ of finite dimensional subspaces, whose union is dense in B, such that each E_α is the range of a projection of norm one from B. It is rather easy to show that L^p-spaces $(1 \leq p < +\infty)$ are π_1-spaces. The problem becomes more difficult for the space $C(S)$ of all continuous scalar (i.e. real or complex) valued functions on a compact Hausdorff space S, and there it is only solved (affirmatively), as far as we know, for metrizable S (cf. [9]).

If $C(S)$ is a π_1-space, then (cf. the footnote on p. 197) each of the spaces E_α appearing in the definition of a π_1-space is isometrically isomorphic to some $l^\infty_{n(\alpha)}$, where l_n^∞ denotes the space of n-tuples of scalars with norm $\|x\| = \max_{1 \leq i \leq n} |x(i)|$. This suggests the following concept.

DEFINITION. A Banach space B is a *π_1^∞-space* provided B has a directed (by inclusion) family of subspaces $(E_\alpha)_{\alpha \in A}$, whose union is dense in B, such that each E_α is isometrically isomorphic to some $l^\infty_{n(\alpha)}$.

It is well known that there always exists a projection from a Banach space onto any subspace which is isometrically isomorphic to some l_n^∞ (see Lemma 2.1 for a proof). Hence every π_1^∞-space is a π_1-space. However, there are π_1-spaces which are not π_1^∞-spaces, such as any Hilbert space (of dimension > 1). In fact, every infinite-dimensional π_1^∞-space is non-reflexive [7; p. 66 Corollary 1, and Theorem 6.1 (2)].

Our principal result asserts that, for separable Banach spaces, the property of being a π_1^∞-space is equivalent to a property (a^∞) which is formally weaker

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and much easier to verify. (If $z \in B$ and $E \subset B$, then $d(z, E)$ will denote $\inf_{e \in E} \| z - e \|$).

(a^{∞}) If Z is a finite subset of B and $\eta > 0$, then there is an integer n and a linear map $T: l_n^{\infty} \to B$ such that $d(z, Tl_n^{\infty}) < \eta$ for $z \in Z$, and

$$(1 + \eta)^{-1} \| x \| \leqq \| Tx \| \leqq (1 + \eta) \| x \|$$

for $x \in l_n^{\infty}$.

THEOREM 1.1. *Let B be a separable Banach space. Then the following conditions are equivalent.*

(i) *B has property (a^{∞}).*

(ii) *B is a π_1^{∞}-space.*

(iii) *B has an increasing sequence of subspaces $E_1 \subset E_2 \subset \cdots$, whose union is dense in B, such that each E_n is isometrically isomorphic to l_n^{∞}.*

This result easily implies (cf. section 6).

COROLLARY 1.2. *Every separable π_1^{∞}-space has a monotone basis.*

Corollary 1.2 improves a result of Gurariĭ [6, Theorem 8], who proved that a separable Banach space with property (a^{∞}) has a basis(*).

Using the standard technique of "peaked partitions of unity" (see Section 5), one can easily verify that $C(S)$ has property (a^{∞}) for arbitrary compact S (cf. [7, pp. 28–29], [9], [11]). Therefore Theorem 1.1 gives an alternative proof of the result [9] that, if S is compact metric, then $C(S)$ is a π_1^{∞}-space. Combining this result with Corollary 1.2 we get the following corollary, which was asserted by Gurariĭ in [6; footnote on p. 298], and which strengthens some results of Vaher [12] and Bessaga [1].

COROLLARY 1.3. *If S is compact metric, then $C(S)$ has a monotone basis.*

In conclusion, let us list, without proof, some further examples of π_1^{∞}-spaces:

The space $C_0(S) = \{f \in C(S): f(x_0) = 0\}$, with S compact metric and $x_0 \in S$.

The spaces $C_{\sigma}(K)$, with K compact metric, of Day [3, p. 89].

The weak tensor product (cf. [3, p. 651]) of any two π_1^{∞}-spaces.

2. **Preliminaries.** We commit the notational abuse of writing $i \in n$ instead of $i = 1, 2, \cdots, n$. If $x = (x(i))_{i \in n} \in l_n^{\infty}$, then

$$N(x) = \{ i \in n : | x(i) | = \| x \| \}.$$

By $u_k^{(n)}$ we denote the k-th unit vector of l_n^{∞}, i.e. $u_k^{(n)}(i) = \delta_k^i$ ($i \in n$, $k \in n$).

In the sequel we shall need the following three lemmas. Lemma 2.2 follows from [4, p. 74, problem 34]. Lemmas 2.1 and 2.3 are probably also known, but we include their proofs for completeness.

(*) Actually, Gurariĭ assumed that B has a property ("B is a space of class $\mathfrak{C}$") which is formally stronger than (a^{∞}). In fact, his property is equivalent to (a^{∞})(cf. [7, p. 22]).

LEMMA 2.1. *Let R be a linear map from l_n^∞ into a Banach space B, and let R have a bounded inverse R^{-1}. Then there exists a projection P from B onto Rl_n^∞ with $\| P \| \leqq \| R \| \| R^{-1} \|$. In particular, if R is an isometry, then $\| P \| = 1$.*

Proof. By the Hahn-Banach theorem, there exist linear functionals ϕ_i on B $(i = 1, \cdots, n)$ such that $\phi_i(y) = (R^{-1}y)(i)$ for all $y \in Rl_n^\infty$, and $\| \phi_i \| \leqq \| R^{-1} \|$. Now let $Px = \sum_{i \in n} \phi_i(x) Ru_i^{(n)}$ for all $x \in B$. Then P is the required projection.

Let $T: l_m^\infty \to l_n^\infty$ be a linear map, and let $e_j = Tu_j^{(m)}$ for $j \in m$.

LEMMA 2.2. $\| T \| = \max_{i \in n} | \sum_{j \in m} e_j(i) |$.

LEMMA 2.3. *T is an isometric embedding if and only if*
(a) $\sum_{j \in m} | e_j(i) | \leqq 1$ *for* $i \in m$,
(b) $\| e_j \| = 1$ *for* $j \in m$,
(c) *if* $i \in N(e_j)$ *and* $s \neq j$, *then* $e_s(i) = 0$ $(s \in m,\ j \in m)$.

Proof. Necessity: The necessity of (a) (by Lemma 2.2) and (b) is trivial. If T is an isometry, $i \in N(e_j)$ and $s \neq j$, then, choosing λ such that $|\lambda| = 1$ and

$$| e_j(i) + \lambda e_s(i) | = | e_j(i) | + | \lambda e_s(i) | = 1 + | e_s(i) |,$$

we get $1 = \| e_j + \lambda e_s \| \geqq 1 + | e_s(i) |$. Hence $e_s(i) = 0$.

Sufficiency: Choose $i_\nu \in N(e_\nu)$ for $\nu \in m$. Let $x = \sum t_j u_j^{(m)} \in l_m^\infty$. By (a) and Lemma 2.2, $\| Tx \| \leqq \| x \|$. On the other hand,

$$\| Tx \| = \max_{i \in n} | \sum_{j \in m} t_j e_j(i) | \geqq \max_{\nu \in m} | \sum_{j \in m} t_j e_j(i_\nu) | = \max_{\nu \in m} | t_\nu | = \| x \|,$$

because, by (b) and (c), $| \sum_{j \in m} t_j e_j(i_\nu) | = | t_\nu |$ for $\nu \in m$. Thus $\| Tx \| = \| x \|$ for $x \in l_m^\infty$.

3. Isometric and almost isometric subspaces of l_n^∞.

LEMMA 3.1. *Let $0 < \eta < 1$. Then, for each linear map $T: l_m^\infty \to l_n^\infty$ such that*

$$(☆) \qquad \| x \| \leqq \| Tx \| \leqq (1 + \eta) \| x \| \qquad \text{for } x \in l_m^\infty,$$

there exists a linear isometric embedding $S: l_m^\infty \to l_n^\infty$ such that $\| S - T \| < \eta$.

Proof. For each $j \in m$, let $f_j = Tu_j^{(m)}$ and

$$N_j = \{ i \in n : | f_j(i) | \geqq 1 \}.$$

By the left-hand side of (☆), $1 = \| u_j^{(m)} \| \leqq \| f_j \|$. Therefore for each $j \in m$ there is an $i_j \in n$ such that $| f_j(i_j) | = \| f_j \| \geqq 1$. Thus the sets N_j are non-empty. They are also mutually disjoint: Indeed, by the right-hand side of (☆) and by Lemma 2.2,

$$1 + \eta \geqq \| T \| \geqq \sum_{j \in m} | f_j(i) | \qquad \text{for } i \in n.$$

Thus if $s \neq j$ ($s \in m, j \in m$), and $i \in N_j$ (or equivalently $|f_j(i)| \geqq 1$), then

$$|f_s(i)| \leqq 1 + \eta - |f_j(i)| \leqq \eta < 1.$$

Hence $i \notin N_s$.

For all $j \in m$ and $i \in n$, let

$$e_j(i) = \begin{cases} f_j(i) \,|\, (f_j(i)|^{-1} & \text{if } i \in N_j, \\ 0 & \text{if } i \in N_s \text{ for } s \neq j \quad (s \in m), \\ f_j(i)(1+\eta)^{-1} & \text{otherwise.} \end{cases}$$

Clearly $\Sigma_{j \in m} |e_j(i)| \leqq 1$ for $i \in n$, $\| e_j \| = 1$ for $j \in m$ and the sets $N(e_j) = N_j$ satisfy condition (c) of Lemma 2.3.

Define the linear map $S: l_m^\infty \to l_n^\infty$ by

$$Sx = \sum_{j \in m} x(j) e_j.$$

Clearly S is an isometric embedding. Since $Tu_j^{(m)} = f_j$ and $Su_j^{(m)} = e_j$, the inequality $\| T - S \| < \eta$ will follow from

$$\sum_{j \in m} |f_j(i) - e_j(i)| < \eta \qquad \text{for } i \in n.$$

We consider two cases. If $i \notin \bigcup_{s \in m} N_s$, then

$$\sum_{j \in m} |f_j(i) - e_j(i)| = \sum_{j \in m} |f_j(i)| \frac{\eta}{1+\eta} < \eta.$$

If $i \in N_s$ for some $s \in m$, then

$$\sum_{j \in m} |f_j(i) - e_j(i)| = \sum_{\substack{j \in m \\ j \neq s}} |f_j(i)| + |f_s(i)| - 1 \leqq \| T \| - 1 < \eta.$$

That completes the proof.

LEMMA 3.2. *If the subspace E of l_n^∞ is isometrically isomorphic to l_m^∞ $(m < n)$, then there is a subspace $F \supset E$ of l_n^∞ which is isometrically isomorphic to l_{m+1}^∞.*

Proof. For $j \in m$, let e_j be the image of the jth unit vector under some fixed isometric isomorphism from l_m^∞ onto E. Therefore the elements e_j satisfy conditions (a), (b) and (c) of Lemma 2.3. Since $m < n$, (c) implies that either one of the sets $N(e_j)$ $(j \in m)$ contains at least two indices, or there is an index $i \in n$ which does not belong to any $N(e_j)$. In both cases it is easy to choose $i_0 \in n$ such that the sets $N(e_j) \setminus \{i_0\}$ are non-empty. Let F be the linear subspace of l_n^∞ spanned by elements f_j $(j \in m + 1)$, where

$$f_j = e_j - e_j(i_0)u_{i_0}^{(n)} \quad \text{for } j \in m,$$
$$f_{m+1} = u_{i_0}^{(n)}.$$

Define $T: l_m^\infty \to l_n^\infty$ by $T(x) = \sum_{j \in m} x(j) f_j$. Then T satisfies conditions (a)–(c) of Lemma 2.3. (We have $N(f_j) = N(e_j) \setminus \{i_0\}$ for $j \in m$, and $N(f_{m+1}) = \{i_0\}$.) Thus F is isometrically isomorphic to l_{m+1}^∞. Finally $F \supset E$ because $e_j = f_j + e_j(i_0) f_{m+1}$ for $j \in m$.

REMARK 3.3. In the language of affine geometry, Lemma 3.2 can be restated as follows:

Let W be an n-dimensional parallelopiped in n-dimensional Euclidean space, and let the origin be the center of symmetry of W. Let L_m be an m-dimensional hyperplane passing through the origin and such that $L_m \cap W$ is an m-dimensional parallelepiped. Then there is an $(m+1)$-dimensional hyperplane L_{m+1} such that $L_{m+1} \supset L_m$ and $L_{m+1} \cap W$ is a parallelepiped.

Some interesting and far-reaching improvements of this result have recently been obtained by Perles.

4. Proof of Theorem 1.1.

LEMMA 4.1. *Let B be a Banach space and let $0 < \eta < 1/6$. Then there is $\sigma(\eta) = \sigma > 0$ such that, if $Q: l_m^\infty \to B$ and $R: l_n^\infty \to B$ are linear maps, and if*

$$(1+\eta)^{-1}\|x\| \leqq \|Qx\| \leqq (1+\eta)\|x\| \qquad \textit{for } x \in l_m^\infty,$$
$$(1+\sigma)^{-1}\|y\| = \|Ry\| \leqq (1+\sigma)\|y\| \qquad \textit{for } y \in l_n^\infty,$$
$$d(Qx, Rl_n^\infty) < \sigma\|x\| \qquad \textit{for } 0 \neq x \in l_m^\infty,$$

then there is an isometric embedding $S: l_m^\infty \to l_n^\infty$ such that $\|RS - Q\| < 12\eta$.

Proof. Choose $\sigma \leqq 1/8\eta$. Let $P: B \to Rl_n^\infty$ be the projection defined in 1.3. Then $\|P\| \leqq \|R\|\,\|R^{-1}\| \leqq (1+\sigma)^2 < 1 + 3\sigma$. Fix $x \in l_m^\infty$ and choose b in Rl_n^∞ such that $\|Qx - b\| < \sigma\|x\|$. Since $Pb = b$, we get

$$\|Qx - PQx\| \leqq \|Qx - b\| + \|Pb - PQx\| \leqq (1 + \|P\|)\sigma\|x\| \leqq 4\sigma\|x\|.$$

Thus

$$\|PQx\| \geqq \|Qx\| - 4\sigma\|x\| \geqq [(1+\eta)^{-1} - \frac{1}{2}\eta]\,\|x\| \geqq (1 - \frac{3}{2}\eta)\|x\|.$$

Hence, using the inequalities $8\sigma < \eta$ and $0 < \eta < 1/4$, we get

$$\|R^{-1}PQx\| \geqq (1+\sigma)^{-1}\|PQx\| \geqq (1 - 2\eta)\|x\|.$$

Let us set

$$T = (1-2\eta)^{-1}R^{-1}PQ.$$

Then, since $\|R^{-1}\|\,\|P\|\,\|Q\| \leqq (1+\sigma)^3(1+\eta) < 1+2\eta$,

$$\|x\| \leqq \|Tx\| \leqq \frac{1+2\eta}{1-2\eta}\|x\| \leqq (1+6\eta)\|x\|.$$

Thus, by Lemma 3.1, there is an isometric embedding $S: l_m^\infty \to l_n^\infty$ such that $\|S-T\| < 6\eta$. Hence

$$\begin{aligned}\|RS-Q\| &\leqq \|R\|\,\|S-T\| + \|RT-Q\| \\ &< (1+\eta)6\eta + \|PQ-Q\| + \|PQ\|\,|1-(1-2\eta)^{-1}| \\ &\leqq 7\eta + 4\sigma + (1+3\sigma)(1+\eta)\frac{2\eta}{1-2\eta} \leqq 12\eta.\end{aligned}$$

That completes the proof.

Proof of Theorem 1.1.

(i)$\to$(ii). First observe that, if B has property (a^∞), then, for any finite dimensional subspace E of B and $\eta > 0$, there is a linear map $T: l_n^\infty \to B$ such that $(1+\eta)^{-1}\|x\| \leqq \|Tx\| \leqq (1+\eta)\|x\|$ for $x \in l_n^\infty$ and $d(e, Tl_n^\infty) < \eta\|e\|$ for $0 \neq e \in E$. To see that, let Z be an $\eta/3$-net for the unit sphere of E, and take $T: l_n^\infty \to B$ as in the definition of (a^∞) with η replaced by $\eta/3$.

Let (b_n) be a countable dense subset of B. By the preceding observation and by Lemma 4.1, we can inductively define, for $\nu = 1, 2, \cdots$, linear maps $T_\nu: l_{n_\nu}^\infty \to B$ and linear isometries $S_\nu: l_{n_\nu}^\infty \to l_{n_{\nu+1}}^\infty$ such that

$$\|T_{\nu+1}S_\nu - T_\nu\| < 12 \cdot 2^{-3\nu},$$
$$\max(\|T_\nu\|, \|T_\nu^{-1}\|) < 1 + 2^{-3\nu},$$

and

$$d(f, T_{\nu+1}l_{n_{\nu+1}}^\infty) < 2^{-3\nu}\|f\| \qquad \text{for } f \in F_\nu,$$

where F_ν is the smallest linear subspace which contains $T_\nu l_{n_\nu}^\infty$ and the elements $b_1, b_2, \cdots, b_\nu$.

For $k = 1, 2, \cdots$ and $\nu = k+1, k+2, \cdots$, let

$$U_{\nu,k} = T_\nu \cdot S_{\nu-1} \cdot S_{\nu-2} \cdots S_k: l_{n_k}^\infty \to B.$$

Since all S_ν are isometric embeddings,

$$\|U_{\nu+1,k} - U_{\nu,k}\| \leqq \|T_{\nu+1}S_\nu - T_\nu\| < 12 \cdot 2^{-3\nu}.$$

Therefore, for all k, the sequence $(U_{\nu,k})_{\nu=k+1}^\infty$ satisfies the Cauchy condition in the operator norm. Let us set

$$U_{\infty,k} = \lim_v U_{v,k},\ E'_k = U_{\infty,k} l^\infty_{n_k} \qquad (k = 1,2,\cdots).$$

Since the S_v are isometric embeddings and since $\lim_v \| T_v \| = \lim_v \| T_v^{-1} \| = 1$, we have

$$\lim_v \| U_{v,k} \| = \lim_v \| U^{-1}_{v,k} \| = \| U_{\infty,k} \| = \| U^{-1}_{\infty,k} \| = 1.$$

Hence the $U_{\infty,k}$ are isometric embeddings, and the E'_k are isometrically isomorphic to $l^\infty_{n_k}(k = 1,2,\cdots)$. Clearly $E'_k \subset E'_{k+1}$, because

$$U_{v,k+1} l^\infty_{n_{k+1}} \supset U_{v,k} l^\infty_{n_k} \quad (k = 1,2,\cdots; v = k+1, k+2,\cdots).$$

Finally we will show that $\bigcup_{k=1}^\infty E'_k$ is dense in B. Take an arbitrary element b_m of the sequence $(b_n)_{n=1}^\infty$, fix $k > m$, choose $x \in l^\infty_{n_k}$ such that $\| T_k x - b_m \| \leqq 2^{-3k} \| b_m \|$, and put $e = U_{\infty,k}\, x$. Clearly

$$\| T_k x - U_{\infty,k} x \| = \lim_v \| T_k x - U_{v,k} x \| .$$

Furthermore

$$\begin{aligned} \| T_k x - U_{v,k} x \| &\leqq \| T_k x - U_{k+1,k} x \| + \sum_{j=1}^{v-k-1} \| (U_{k+j+1,k} - U_{k+j,k}) x \| \\ &\leqq \sum_{i=k}^{v-1} 12 \cdot 2^{-3i} \| x \| \leqq \sum_{i=k}^{\infty} 12 \cdot 2^{-3i} \| x \| . \end{aligned}$$

By the definition of x,

$$\| x \| \leqq \| T_k^{-1} \| \cdot \| T_k x \| \leqq \| T_k^{-1} \| (1 + 2^{-3k}) \| b_m \| .$$

Hence

$$\begin{aligned} \| b_m - e \| &\leqq \| T_k x - b_m \| + \| T_k x - U_{\infty,k} x \| \\ &\leqq 2^{-3k} \| b_m \| + \| T_k^{-1} \| (2^{-3k} + 1) \sum_{i=k}^{\infty} 12 \cdot 3^{-i} \| b_m \| . \end{aligned}$$

Since $\lim_k \| T_k^{-1} \| = 1$, the last inequality implies $\lim_k d(b_m, E'_k) = 0$. Since the sequence (b_n) is dense in B, this completes the proof that the union $\bigcup_{k=1}^\infty E'_k$ is dense in B.

(ii) $\rightarrow$ (iii). If B is a separable π_1^∞ space, then there is an increasing sequence $(E'_n)_{n=1}^\infty$ of subspaces of B such that $\bigcup_{n=1}^\infty E'_n$ is dense in B, and E'_k is isometrically isomorphic to $l^\infty_{n_k}$ for some n_k $(k=1,2,\cdots)$. (Indeed, let E_α be as in the definition of a π_1^∞-space, and let $\{b_n\}_{n=1}^\infty$ be a dense subset of B. By induction, choose $E'_n = E_\alpha$ such that $d(b_m, E'_n) < n^{-1}$ for $m = 1,\cdots,n$, and $E'_n \supset E'_{n-1}$.) Therefore, to complete the proof of the implication (ii) $\rightarrow$ (iii), it is enough "fill in the gaps", i.e. to show that, if $k = 1,2,\cdots$ and $n_{k+1} - n_k > 1$, then there is a chain of subspaces $E'_k = F_0 \subset F_1 \subset \cdots F_{n_{k+1}-n_k} = E'_{k+1}$ such that F_v is isometrically isomorphic to

$l^{\infty}_{n_k+\nu}$ $(\nu = 0, 1, 2, \cdots, n_{k+1} - n_k)$. But the existence of such a chain of subspaces follows immediately from Lemma 3.2.

(iii) $\rightarrow$ (i). This implication is trivial.

5. **A refinement of Theorem 1.1 for** $B = C(S)$. In the case of a $C(S)$ space, S compact metric, one can prove a slightly stronger result than Theorem 1.1 (cf. Corollary 5.2 below). We recall that a finite-dimensional subspace E of $C(S)$ is called a *peaked partition subspace* provided it is spanned by a *peaked partition of unity*, i.e. by non-negative functions $f_1, f_2, \cdots, f_n$ such that $\Sigma f_i = \mathbf{1}$ and $\| f_i \| = 1$ $(i = 1, 2, \cdots n)$, where $\mathbf{1}$ denotes the function which is identically one on S.

PROPOSITION 5.1. *Let E be a linear subspace of $C(S)$. Then the following conditions are equivalent.*

(o) *E is a peaked partition subspace.*

(oo) *E is isometrically isomorphic to l^{∞}_n for some n, and $\mathbf{1} \in E$.*

Proof. (o) $\rightarrow$ (oo). This implication is well known (cf. e.g. [9], [11]).

(oo) $\rightarrow$ (o). Let $f'_k \in E$ correspond under some isometric isomorphism to the k-th unit vector $u^{(n)}_k$ of l^{∞}_n for $k \in n$. Then clearly $\| f'_k \| = 1$. Let us set $f_k = [f'_k(s_k)]^{-1} f'_k$, where $s_k \in S$ is chosen in such a way that $|f'_k(s_k)| = 1$. Clearly (cf. Lemma 2.3 (c)) $f_k(s_l) = 0$ for $l \neq k$, $l \in n$. Since $\mathbf{1} \in E$, there are scalars $(t^0_k)_{k \in n}$ such that $\mathbf{1} = \sum^n_{k=1} t^0_k f_k$. Thus $1 = \mathbf{1}(s_l) = \sum^n_{k=1} t^0_k f_k(s_l) = t^0_l$ for $l \in n$. Thus $\sum^{\infty}_{k=1} f_k(s) = 1$ for $s \in S$. Since $\| \sum^n_{k=1} t_k f_k \| = \| \sum^n_{k=1} t_k f'_k \| = \max_{k \in n} |t_k|$ for arbitrary scalars $t_1, t_2, \cdots, t_n$, we get $\sum^n_{k=1} |f_k(s)| \leqq 1$ for $s \in S$. Thus all f_k are non-negative. Hence $\{f_1, f_2, \cdots, f_n\}$ is a peaked partition of unity.

Combining Proposition 5.1 with the main result of [9] and Lemma 3.2 (cf. the proof of implication (ii) $\rightarrow$ (iii)) we get

COROLLARY 5.2. *Let E be an m-dimensional peaked partition subspace in $C(S)$, S compact metric. Then there exists an increasing sequence of peaked partition subspaces $E_1 \subset E_2 \subset \cdots$ of $C(S)$, whose union is dense in $C(S)$, such that* $\dim E_n = n$ *for all n and $E_m = E$.*

6. **Monotone bases in separable π^{∞}_1-spaces.** We recall (cf. [3, p. 67]) that a sequence $(e_n)^{\infty}_{n=1}$ is called a *(monotone) basis* for a Banach space B provided each b in B has a unique expansion $b = \sum^{\infty}_{i=1} t_i e_i$ (and $\| b \| \geqq \| \sum^n_{i=1} t_i e_i \|$ *for* $n = 1, 2, \cdots$). If $(e_n)^{\infty}_{n=1}$ is a monotone basis for B, then the operator $P_n b = \sum^n_{i=1} t_i e_i$, for $b = \sum^{\infty}_{i=1} t_i e_i \in B$, is a projection of norm one from B onto the subspace E_n spanned by $e_1, e_2, \cdots, e_n$. Therefore the existence of a monotone basis in B implies the existence of projections $P_n : B \rightarrow E_n$ such that

(α) $\| P_n \| = 1$,

(β) each range $P_n B = E_n$ is an n-dimensional subspace of B,

(γ) $E_n \subset E_{n+1}$ $\qquad (n = 1, 2, \cdots)$,

(δ) $E = \bigcup^{\infty}_{n=1} E_n$ is dense in B.

Conversely, the following observation is due to S. Mazur (cf. Bessaga [2]).

PROPOSITION 6.1. *If in Banach space B there exists a sequence of projections $(P_n)_{n=1}^{\infty}$ satisfying conditions (α)–(δ), then B has a monotone basis.*

Proof. Define $(e_n)_{n=1}^{\infty}$ inductively such that $\|e_n\| = 1$ and $e_n \in E_n \cap \ker P_{n-1}$ for $n = 1,2,\cdots$. (For convenience, we set $P_0 = 0$.) Since the range of P_{n-1} is $(n-1)$-dimensional, the kernel $\operatorname{Ker} P_{n-1}$ has codimension $= n-1$. Therefore the intersection $E_n \cap \operatorname{Ker} P_{n-1}$ is non-empty. To prove that $(e_n)_{n=1}^{\infty}$ is a monotone basis for B, observe first that, since $\|P_n\| = 1$, $e_{n+1} \in \operatorname{Ker} P_n$ and $e_\nu \in E_n$ for $\nu = 1,2,\cdots,n$.

$$\left\|\sum_{\nu=1}^{n+1} t_\nu e_\nu\right\| \geqq \left\|P_n\left(\sum_{\nu=1}^{n+1} t_\nu e_\nu\right)\right\| = \left\|\sum_{\nu=1}^{n} t_\nu e_\nu\right\|$$

for arbitrary scalars $t_1, t_2, \cdots, t_{n+1}$. Thus by induction $\|\sum_{\nu=1}^{n+m} t_\nu e_\nu\| \geqq \|\sum_{\nu=1}^{n} t_\nu e_\nu\|$ for arbitrary scalars $t_1, t_2, \cdots, t_{n+m}$ $(n, m = 1,2,\cdots)$. But the last inequality, together with (δ), implies that $(e_n)_{n=1}^{\infty}$ is a monotone basis for B. (c. f. [10])

Proof of Corollary 1.2. This follows from Theorem 1.1, Lemma 2.1 and Proposition 6.1.

It follows from a result of Lindenstrauss [7] that, if B is a π_1^∞-space and if E is the range of a projection of norm one from B with dim $E = n < +\infty$, then E is isometrically isomorphic to l_n^∞(*). Hence we can complete Corollary 1.2 as follows:

COROLLARY 6.2. *Let B together with a sequence of subspaces (E_n) satisfy condition* (iii) *of Theorem* 1.1. *Then there exists in B a monotone basis $(e_n)_{n=1}^{\infty}$ such that $\{e_1, e_2, \cdots, e_n\}$ spans E_n $(n = 1,2,\cdots)$. Conversely, if $(e_n)_{n=1}^{\infty}$ is a monotone basis for a π_1^∞-space B, then B and the subspaces E_n spanned by $\{e_1, e_2, \cdots, e_n\}$ satisfy condition* (iii) *of Theorem* 1.1.

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(*) Indeed, if B is a π_1^∞-space, then, by Corollary 1 of [7, p. 66], B satisfies conditions (1)–(13) of Theorem 6.1 of [7, p. 62]. Since E is the range of a projection of norm one from B, the subspace E also satisfies the same conditions. In particular, E^{**} is a $\mathscr{P}_1$-space. Since E is finite-dimensional, $E^{**} = E$. Thus E is isometrically isomorphic to l_n^∞.

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